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THE SPHERICAL RAMAN-NATH EQUATION WITH TIME-DEPENDENT COEFFICIENTS

G. Dattoli and J. Gallardo

Lettere al Nuovo Cimento
1985

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THE SPHERICAL RAMAN-NATH EQUATION WITH TIME-DEPENDENT COEFFICIENTS

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ABSTRACT.

In this letter we write the exact solution of the Spherical Raman-Nath equation with time dependent coefficients in terms of a hypergeometric function. Finally we indicate how already known results can be derived from this more general expression.

The Raman-Nath (R.N.) type equations are a set of differential recursive equations (1), which have become familiar to the physics community in recent times. They are indeed relevant to a large number of physical problems; to quote a few of them we recall light diffraction by ultrasounds (2), atomic and molecular dynamics (3), and Free Electron Lasers (4). (For a more extended list, the reader is referred to Ref. 1.)

A general and systematic analysis of these equations has been put forward in Ref. (1,5). In these papers an operational technique, helpful to look for both exact and non trivial perturbed solutions, has been developed.

Together with the R.N. equations discussed in Refs. (1,5), a more general class of recursive differential equations has been considered(6).These equations, which appear in the analysis of stimulated Compton-Scattering (4) or may be exploited to write the time evolution of the so called coherent Block states (7-9), has been named spherical or SU2-R.N. equations .

In this note we will find a solution for the most general form of SU2-R.N. equation, solvable in terms of known functions, and we will indicate how all the previously found solutions (1,5,6) are only particular cases of this more general expression.

The equation we consider is the following :

$$i \frac{d}{d\tau} C_\ell(\tau) = \omega(\tau) \left(\frac{n_+ - n_-}{2} + \ell \right) C_\ell(\tau) + \Omega(\tau) \sqrt{(n_- - \ell)(n_+ + \ell + 1)}\, C_{\ell+1}(\tau)$$
$$+ \Omega^*(\tau) \sqrt{(n_+ + \ell)(n_- - \ell + 1)}\, C_{\ell-1}(\tau)$$
$$C_\ell(0) = \delta_{\ell,0} \qquad (1)$$

where $n_\pm$ are fixed integers, $\omega(t)$ and $\Omega(t)$ are time-dependent integrable functions, and ℓ is a discrete variable ($-n_- \leq \ell \leq n_+$).

We impose the initial condition $c_\ell(0) = \delta_{\ell,0}$. This choice is not restrictive and indeed for general initial condition of the form $c_\ell(0) = f_\ell$ the solution of Eq.(1) can be obtained following the procedure below with only very few changes (5).

The technique, adopted to find the solution of the above equation, is a generalization of that presented in Ref. (5). We rewrite Eq. (1) in a more compact form by defining

$$C_\ell(\tau) = (-i)^\ell \exp\left[-i \left(\frac{n_+ - n_-}{2} + \ell \right) \int_0^\tau d\tau' \, \omega(\tau') \right] M_\ell(\tau) \qquad (2)$$

Introducing angular momentum-type operators, we easily find

$$\frac{d}{d\tau} M_{\ell}(\tau) = -\Omega(\tau) \exp\left[-i\int_0^{\tau} d\tau' \omega(\tau')\right] \hat{J}_+ M_{\ell}(\tau) + \Omega^*(\tau) \exp\left[i\int_0^{\tau} d\tau' \omega(\tau')\right] \hat{J}_- M_{\ell}(\tau) \tag{3}$$

The J operators "act" on M_{ℓ} according to the rules

$$\begin{aligned} \hat{J}_3 M_{\ell}(\tau) &= \left(\frac{n_+ - n_-}{2} + \ell\right) M_{\ell}(\tau) \\ \hat{J}_{\pm} M_{\ell}(\tau) &= \sqrt{(n_{\mp} \mp \ell)(n_{\pm} \pm \ell + 1)}\, M_{\ell \pm 1}(\tau) \end{aligned} \tag{4}$$

They obey the well-known rules of commutation

$$\left[\hat{J}_+, \hat{J}_-\right] = 2\hat{J}_3 \quad , \quad \left[\hat{J}_3, \hat{J}_{\pm}\right] = \pm \hat{J}_{\pm} \tag{5}$$

The use of the above operators clarifies the term "SU2- R.N. equation" to indicate those equations which can be cast in the general form of Eq.(1).

The key-point in our derivation will be the use of the so called Wie-Norman Lie-algebraic analysis (10) of linear differential equations. We can, therefore, bypass all the problems relevant to the time ordering and coupling exponentials to write

$$M_{\ell}(\tau) = e^{2h(\tau)\hat{J}_3}\, e^{-g(\tau)\hat{J}_+}\, e^{f(\tau)\hat{J}_-}\, i^{\ell}\, \delta_{\ell,0} \tag{6}$$

where the functions h(t), g(t), and f(t) obey the following system of differential equations

$$\frac{d}{d\tau}h(\tau) = g(\tau)\frac{d}{d\tau}f(\tau)$$
$$\frac{d}{d\tau}g(\tau) = \Omega(\tau)\exp\left[-2h(\tau) - i\int_0^\tau d\tau'\,\omega(\tau')\right] - g(\tau)\frac{d}{d\tau}h(\tau)$$
$$\frac{d}{d\tau}f(\tau) = \Omega^*(\tau)\exp\left[2h(\tau) + i\int_0^\tau d\tau'\,\omega(\tau')\right] \qquad (7)$$

with initial conditions, h(0)=g(0)=f(0)=0. This system can be solved if the solution of the following Riccati equation is known

$$\frac{d}{d\tau}u(\tau) - u^2(\tau) + p(\tau)\,u(\tau) + q(\tau) = 0$$
$$u(0) = 0 \qquad (8)$$

where we have set

$$u(\tau) = \frac{d}{d\tau}h(\tau)$$
$$p(\tau) = -i\,\omega(\tau) - \left[\frac{1}{\Omega(\tau)}\frac{d}{d\tau}\Omega(\tau)\right]^* \qquad (9)$$
$$q(\tau) = -\left|\Omega(\tau)\right|^2$$

However, even if we do not know the explicit solution of the Eq.(8), we can write down the expression of $C_\ell(\tau)$ in terms of the functions $f(\tau)$, $g(\tau)$, $h(\tau)$. This can be accomplished in a rather direct way. We use a series expansion

of the exponentials and the operational rules Eq.(4), so that, after a lengthy algebra we find

$$C_\ell(\tau) = \sqrt{\binom{n_-}{\ell}\binom{n_+ + \ell}{\ell}} \exp\left[\frac{n_- - n_+}{2}\mathcal{H}(\tau)\right]$$
$$\times \left(\mathcal{F}(\tau)\right)^{\ell} {}_2F_1\left(-n_+, n_- + 1; \ell + 1; \mathcal{F}(\tau)\,\mathcal{G}(\tau)\right) \tag{10}$$

where to simplify the notation we have defined

$$\begin{aligned} \mathcal{H}(\tau) &= i\int_0^\tau d\tau'\,\omega(\tau') - 2\,h(\tau) \\ \mathcal{F}(\tau) &= -i\,f(\tau)\,\exp\left[-i\int_0^\tau d\tau'\,\omega(\tau')\right] \\ \mathcal{G}(\tau) &= i\,g(\tau)\,\exp\left[i\int_0^\tau d\tau'\,\omega(\tau')\right] \end{aligned} \tag{11}$$

also ${}_2F_1(a,b;c;z)$ is the hypergeometric function (11).

The result in Eq.(10) is a very general one and it allows us to calculate the probability amplitude for the stimulated Compton scattering (4). Furthermore, it may be exploited to recover all previously known solutions (1,5,6).

If we take $\Omega(\tau)$ and $\omega(\tau)$ as time independent functions and $n_+ = 0$, it is trivial to recover the solution written in Ref. (6) in terms of the binomial generalization of the Poisson-

Charlier function(12).

The connection with the cases discussed in Refs. (1,5) requires the use of the symptotic properties of ${}_2F_1$ (a,b;c;z); taking the large n- limit the hypergeometric function reduces to a Laguerre function (11) so that Eq.(10) yields the solution of the Harmonic R.N. equation . Finally taking both n_+ and n_- very large , ${}_2F_1$ (a,b;c;z) becomes a Bessel function (11) and thus we recover the solution of the "shifted" R.N. equation .

In summary, we have found an exact solution of the SU2-R.N. equation. In a forthcoming paper we will apply this solution to the analysis of the time evolution of the coherent Block-states(9). In that paper we will discuss, in more details, the problem of the asymptotic limits touched upon in the last part of this letter.

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